Journal of Agricultural and Resource Economics 36(1):95–120

The Economics of Commonly Owned Groundwater When User Demand Is Perfectly Inelastic

Chenggang Wang and Eduardo Segarra

This paper provides a theoretical analysis of the common-pool resource dilemma in extracting nonrenewable groundwater resources when water demand is perfectly inelastic. It complements the existing theory of groundwater use, which assumes away the possibility of demand perfect inelasticity. Under perfectly inelastic water demand, the common-pool resource dilemma is by-passed if groundwater users are equally productive in water use. If they are not, a new type of inefficiency can arise due to the lack of a rationing mechanism on the basis of productivity. Our analysis suggests that groundwater management research should pay more attention to water demand elasticity and productivity heterogeneity.

Key Words: common-pool resources, groundwater extraction, groundwater management, groundwater use, irrigation water demand, optimal control, pumping externality, water demand elasticity

Introduction

Economists have long viewed groundwater as a classic example of a common-pool resource (CPR), where competitive or open-access exploitation is inefficient. Although land ownership precludes free entry to a groundwater basin, it is difficult to establish property rights to groundwater units. Stock externalities arise because one groundwater user's withdrawal increases the pumping lift of all other users. Individual users' failure to account for negative stock externalities leads to overexploitation, and the total economic value of the resource cannot be fully realized. In other words, individual rationality leads to an outcome that is not rational to the group—a behavioral outcome known as the CPR dilemma (Ostrom, Gardner, and Walker, 1994).

Multiple models have been developed to understand the mechanisms of welfare loss caused by stock externalities, the simplest of which assumes a type of myopic behavior by groundwater users. In this scenario, each user determines the rate of extraction by maximizing his or her own current benefits, taking no account of that decision's impact on the future pumping costs of all users, including his or her own (e.g., Allen and Gisser, 1984; Feinerman and Knapp, 1983; Gisser and Sanchez, 1980; Nieswiadomy, 1985; Worthington, Burt, and Brustkern, 1985).

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Review coordinated by Myles J. Watts.

Another class of models considers the problem as a dynamic game played by groundwater users. Two distinct equilibrium concepts have emerged in the literature to solve the game. In Nash equilibrium, each groundwater user decides at the beginning of the game the entire path of extraction that maximizes the discounted present value of the benefits derived from extracted groundwater, in the belief others will behave similarly. Unlike the myopic strategy, the Nash strategy postulates that groundwater users take into account stock externalities, but it also assumes groundwater users will commit to that strategy once the optimal path of extraction is determined. This assumption has been criticized as strong and unstable. Dixon (1988), Negri (1989), and Provencher and Burt (1993) propose an alternative equilibrium, the Markov-Nash, which replaces the optimal path of extraction in the Nash equilibrium with an optimal decision rule that depends only on the present resource stock. Based on findings reported by Provencher and Burt, the myopic strategy leads to an outcome that is farthest away from the socially optimal outcome, and between the two outcomes lie the Nash and Markov-Nash strategies.

Together with the growth of the theoretical literature, an empirical literature has developed which aims to quantitatively compute welfare losses arising from competitive extraction. Gisser and colleagues (Allen and Gisser, 1984; Gisser and Sanchez, 1980) find that socially optimal extraction offers negligible gains over the competitive myopic strategy in the Pecos Basin of New Mexico. Research conducted by Gisser and his colleagues inspired a number of similar studies on other groundwater basins, some of which relaxed restrictive assumptions in the Gisser and Sanchez model. Koundouri (2004a, b) provides a comprehensive and insightful review of this literature, concluding that the magnitude of the welfare loss caused by competitive exploitation varies from one basin to another, depending on a variety of economic, hydrologic, and agronomic factors. The loss is particularly sensitive to the slope of water demand and moderately sensitive to aquifer storage capacity. Specifically, the loss tends to be small when the slope of the water demand curve is close to infinity (meaning demand is highly irresponsive to price changes) or when aquifer storage capacity is large. Additionally, welfare loss may be large in the presence of large heterogeneity in groundwater users' productivity.

Implicit in the groundwater extraction models reviewed above is a downward-sloping, nonvertical water demand curve; i.e., demand will always respond to price changes. This assumption is directly imposed in models built on a water demand function. Models that start with a production function often impose strict concavity on that function, which leads, under profit-maximization behavior, to a downward-sloping, nonvertical input demand. Demand perfect inelasticity is assumed away as a matter of convenience: It warrants the existence of an interior smooth solution path that is amenable to comparative dynamic analysis. However, neither theoretical reasoning nor empirical evidence justifies the exclusion of perfect inelasticity as an anomaly rarely occurring in the real world. Crop scientists and irrigation engineers have found a linear relationship between evapotranspiration and yield for most crops (Doorenbos and Kassam, 1979). If we assume yield will stop responding to water after reaching the plant's varietal potential or other agronomic factors become limiting, as indicated by the law of the minimum (Paris, 1992), the linear water-yield relation implies a profit-maximizing irrigator's water demand is perfectly inelastic.[1]

Empirical evidence abounds that water demand is highly inelastic, especially at low price levels (Berbel and Gómez-Limón, 2000; Gardner and Young, 1984; Gisser et al., 1979; Heady

[1] A more detailed discussion of the technological structure underlying a perfectly inelastic water demand is provided in the next section.

et al., 1973; Moore and Hedges, 1963; Scheierling, Young, and Cardon, 2004). Using data from an extensive survey of irrigation farmers in the western United States, Moore, Gollehon, and Carey (1994) examine the water demands of major crops in four western U.S. regions. Their results show the price elasticity of water demand is significantly different from zero at the 10% level only for one of the 20 region-crop combinations examined, though a significant long-run water use response is found through changes in cropping patterns.

Inelastic water demand is also corroborated by the failure of water pricing policies. The notion of using water pricing as a policy instrument to improve irrigation efficiency and consumptive water use has been around for decades, but the impact of water charges on water use has been elusive (Bosworth et al., 2002; Garrido, 2002). Molle and Berkoff (2007) provide a narrative of the frustration policy makers have experienced with using water pricing as a tool for managing water development projects, and of their acknowledgement after years of on-the-ground experience that a large gap exists between water pricing theory and reality.

Thus, one should not conclude that water demand perfect inelasticity is unlikely to occur in the real world. In the short run, the farmer's water demand is likely to be fixed by water availability, the plant's varietal potential, and the established irrigation practice. In the long run, irrigation technology and crop choice, and consequently water demand, can change in response to changes in water price. However, many real-world situations may come to mind where the short run is too long to be ignored. For example, farmers in developed countries who have already adopted sophisticated center pivot or drip irrigation systems have little room for further efficiency improvements. Poor farmers in developing countries often find themselves in a poverty trap, where income remains low precisely because they cannot afford to upgrade irrigation technologies and introduce water-saving varieties. Crop choice may be limited by climatic and hydrologic factors. In the Texas High Plains, where groundwater supply and rainfall are barely able to sustain water-intensive crops such as corn, cotton remains the best crop to grow even in the face of declining cotton prices and rising corn prices.

By assuming away water demand perfect inelasticity, the existing economic theory of groundwater use is unable to offer useful policy guidance for these situations, leaving many challenging questions unanswered. In the face of a perfectly inelastic water demand, for example, what are the optimal extraction strategies for competitive groundwater users when they behave myopically or strategically in the Nash and Nash-Markov senses? Can the CPR dilemma be avoided when water demand is perfectly inelastic? How does heterogeneity affect efficiency of groundwater use?

The purpose of this study is to fill this gap. Our analysis complements the existing theory of groundwater use in irrigated agriculture by providing a comprehensive treatment of the CPR dilemma when water demand is perfectly inelastic. We develop a model of non-renewable groundwater extraction with two farms, which differ in productivity of utilizing groundwater to generate profits. When water demand is perfectly inelastic, and in the absence of productivity heterogeneity, we find competitive exploitation leads to a socially optimal outcome and the CPR dilemma is bypassed, regardless of whether users behave myopically or strategically. In the presence of heterogeneity, however, a suboptimal outcome arises under competitive extraction. The cause of this inefficient outcome is not the stock externalities highlighted in most groundwater models; rather, inefficiency arises because without well-defined, tradable property rights to groundwater units, a coordinating mechanism is lacking to allocate resources on the basis of productivity—resulting in overuse by less productive users and underuse by more productive users.

Our analysis also sheds light on some predictions of previous models. While simulation studies have found the welfare loss arising in competitive extraction can be nontrivial in the presence of heterogeneous users (Koundouri, 2000; Laukkanen and Koundouri, 2006; Worthington, Burt, and Brustkern, 1985), the mechanism of such an effect has not been well understood, due to the analytical difficulty of separating heterogeneity-induced externalities from stock externalities. With stock externalities being purged out by the demand perfect inelasticity assumption, our model permits a clear understanding of the heterogeneity-induced CPR problem. Further, Gisser and Sanchez (1980) have found that welfare loss arising in competitive exploitation of groundwater resources tends to be small for aquifers with large storage capacity. Our analysis reveals this prediction is valid only if groundwater users are equally productive. When they are not, an opposite effect can arise, because larger storage capacity implies more economic value is at stake if groundwater is not efficiently allocated on the basis of productivity.

Water Demand and Model Assumptions

Most dynamic groundwater use models comprise two interconnected systems, one depicting groundwater users' consumption behavior and another illustrating aquifer system dynamics. Two different approaches have been followed to set up the economic subsystem. One approach directly incorporates a water demand function in the model and assumes cost-minimization behavior (e.g., Gisser and Sanchez, 1980). The other starts with a production or benefit function and assumes profit-maximization behavior (e.g., Provencher and Burt, 1993). Here, we take the second approach, because given the practical difficulty of estimating water demand due to the lack of water consumption and price data, it is useful to characterize the possible biological, agronomic, and hydrologic constraints underlying a particular type of demand behavior—in our case, perfect inelasticity.

Crop scientists generally believe the relationship between crop yields and essential nutrients and water is governed by von Liebig's law of the minimum (Havlin et al., 1999), which posits that crop yield is determined by the most deficient production factor, and not by the relatively abundant ones. In particular, crop yields can be increased only by increasing the most deficient factor until some other factor becomes limiting. The key notion in von Liebig's law is the nonsubstitutability of essential inputs for crop growth (Paris, 1992). In a single-factor model, the law is mathematically represented by the Liebig-Paris function:

$$y = \min\{f(x), M\},$$

where y is crop yield in the given field, $f(x)$ is an increasing function of input x, and M is a constant representing the yield plateau determined by other growth factors such as soil quality, climate conditions, and the varietal potential. A special case of the Liebig-Paris function is the linear response and plateau (LRP) function, where $f(x)$ is linear. The LRP model is particularly appropriate for describing yield-water relationships, as a linear relationship has generally been found between evapotranspiration and yield for most crops under a wide range of growing conditions (Doorenbos and Kassam, 1979). Formally, we assume the technology is of the following form:

(1)
$$F(w) = \begin{cases} a + bw & \text{for } w \in [0, W), \\ a + bW & \text{for } w \in [W, +\infty), \end{cases}$$

where w is the amount of water consumed by the crop, a is crop yield corresponding to nonirrigated or dryland farming, b represents the marginal physical product of applied water, and $W > 0$ is the minimal amount of water needed for yield to reach the plateau. In appendix A, we show that the LRP model can be derived from the water-yield response model widely used in the crop irrigation literature and detailed in Doorenbos and Kassam (1979).[2]

Let $\varepsilon \in (0, 1)$ denote irrigation efficiency; i.e., one unit of water pumped out of the aquifer amounts to only ε units of water consumed by the crop. Production function (1) thus can be expressed in terms of the amount of water pumped from the aquifer x:

$$G(x) = \begin{cases} a + b\varepsilon x & \text{for } 0 \le x \le W/\varepsilon, \\ a + bW & \text{for } x > W/\varepsilon. \end{cases} \tag{2}$$

In addition to the restriction of the yield plateau, limited groundwater availability can be a binding constraint to water consumption. In each crop season, groundwater is available only up to the well yield capacity, X, which is determined by certain properties of the aquifer, i.e., $0 \le x \le X$. Denoting by p the price or marginal pumping cost of groundwater normalized by crop price, a profit-maximizing farm's water demand is the solution to the following problem:

$$\max_{0 \le x \le X} G(x) - px.$$

It is straightforward to show that the farm's water demand is:

$$x = \begin{cases} \min\{X, W/\varepsilon\} & \text{if } p < b\varepsilon, \\ [0, \min\{X, W/\varepsilon\}] & \text{if } p = b\varepsilon, \\ 0 & \text{if } p > b\varepsilon. \end{cases} \tag{3}$$

Water demand is perfectly inelastic everywhere except when price is equal to $b\varepsilon$ (to avoid mathematical technicalities, we assume demand is zero when $p = b\varepsilon$ in the remainder of the paper). When price is below this threshold, groundwater demand is the maximal amount allowed by the binding constraints. When price equals or exceeds the threshold, quantity demanded of groundwater becomes zero and dryland farming is adopted.

The Representative-Farm Model

A groundwater user behaves myopically if he or she maximizes her own current-period profits and takes no account of future profits. Under myopic, competitive behavior, by maximizing individual profits, groundwater users maximize their collective profits; thus, replacing individual users' profit-maximization problems with those of a representative user generates no aggregation bias. The representative-user model has been widely used in economic analysis as a shortcut to solving static multi-user problems under perfect competition. The cost of that simplification is losing sight of the effects of heterogeneity (e.g., Gisser and Sanchez, 1980). The optimal control strategy maximizes the discounted present value of all benefits that can be derived from extracted groundwater, representing the most efficient way to manage the common-pool resource by a benevolent social planner.

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[2] The LRP technology assumes away the possibility that overirrigation can reduce yield. This assumption is innocuous in our analysis because a rational irrigator will never overirrigate—overuse of a non-free resource without further increasing output is a suboptimal choice.

Let $\bar{s}$ and s represent the elevations of the farmland surface and water table in the aquifer, respectively. Pumping costs are assumed to be linear in pumping lift, $\bar{s}-s$; that is:

$$P(s,x)=\gamma(\bar{s}-s)x, \tag{4}$$

where γ is the cost of pumping per unit of water per unit of lift, a constant determined by the pump's properties and energy price. Other operating costs are assumed away as they are tangential to the dynamic decision of groundwater extraction.

The dynamics of the water table are governed by the differential equation:

$$\dot{s}(t)=-\kappa x(t), \quad s(0)=s^0, \tag{5}$$

where κ is a parameter that depends negatively on the storativity S and covered area A of the aquifer, i.e., $\kappa = 1/SA$. Storativity represents the volumetric fraction of the aquifer which is occupied with water. As denoted by equation (5), every inch of water pumped out of the aquifer lowers the water table by κ inches. Note that implicit in equation (5) is a zero recharge rate, so we are dealing with a nonrenewable resource.[3]

Under myopic behavior, the instantaneous water consumption as a function of the water table can be obtained by substituting unitary pumping cost $\gamma(\bar{s}-s)$ for water price p in (3):

$$x=\begin{cases}\min\{X, W/\varepsilon\} & \text{if } s>\bar{s}-b\varepsilon/\gamma,\\ 0 & \text{if } s\le\bar{s}-b\varepsilon/\gamma,\end{cases} \tag{6}$$

where $\bar{s}-b\varepsilon/\gamma$ is the threshold water level where the marginal pumping cost and the marginal product value equalize, and where dryland farming and irrigated farming are equally profitable. The threshold water level $\bar{s}-b\varepsilon/\gamma$ can be viewed as an index for the representative farm's productivity when using groundwater to generate profits. It summarizes the technological and hydrological conditions facing a farm, including the elevation of farmland surface $\bar{s}$, marginal product value of groundwater b, irrigation efficiency ε, and unitary pumping cost γ. A lower value of this index means the aquifer can sustain irrigation for a longer period. For example, the index is lower for a farm that grows a higher-value crop, for a farm with a more efficient irrigation system, and for a "valley farm" than for a "hill farm."

If the initial water table is above the threshold (i.e., $s^0>\bar{s}-b\varepsilon/\gamma$), a myopic representative farm will pump at a constant rate equal to the maximal amount allowed by the binding constraints, until the water table declines to the threshold level at which irrigation is abandoned permanently. The path of the water table can be obtained by integrating equation (5):

$$s=\begin{cases}s^0-\kappa t\min\{X, W/\varepsilon\} & \text{if } t<T,\\ \bar{s}-b\varepsilon/\gamma & \text{if } t\ge T,\end{cases} \tag{7}$$

where

$$T=\frac{s^0-(\bar{s}-b\varepsilon/\gamma)}{\kappa\min\{X, W/\varepsilon\}} \tag{8}$$

is the aquifer's usable life for irrigation, which is the time needed for the water table to decline from initial level s^0 to threshold level $\bar{s}-b\varepsilon/\gamma$ at a constant rate of $\kappa\min\{X, W/\varepsilon\}$. If the initial water table is below the threshold (i.e., $s^0<\bar{s}-b\varepsilon/\gamma$), then irrigated farming is

[3] Recharge includes both natural recharge and return flow from irrigation.

less profitable than dryland farming and the latter will be adopted throughout. This implies the groundwater resource is of no value to the farm.

The social planner's problem is to maximize the discounted present value of the farm's profits over an infinite horizon:

(9) $$\int_0^\infty e^{-rt}\left(G\big(x(t)\big) - P\big(s(t), x(t)\big)\right) dt,$$

subject to equations (2), (4), and (5) and the hydrological constraint:

(10) $$0 \le x(t) \le X,$$

where r is the interest rate.

An inspection of the planner's problem reveals that extracting more than W/ε units of water at any point in time generates no extra revenue but raises pumping costs in the future. An optimal path of extraction, therefore, is necessarily bounded above by W/ε. With this in mind, the social planner's problem can be rewritten as:

(P.1) $$\begin{aligned} &\max_{x(\cdot)} \int_0^\infty e^{-rt}\left[a + \left[b\varepsilon - \gamma\left(\bar{s} - s(t)\right)\right]x(t)\right] dt \\ &\text{s.t.:}\ \ \dot{s}(t) = -\kappa x(t), \\ &\qquad s(0) = s^0, \\ &\qquad 0 \le x(t) \le \min\{X, W/\varepsilon\}. \end{aligned}$$

This is a linear optimal control problem with a compact control set, the solution to which is the well-known "bang-bang" solution. The optimal path of extraction must necessarily maximize the current-value Hamiltonian:

(11) $$H = a + \left[b\varepsilon - \gamma(\bar{s} - s) - \kappa\lambda\right]x,$$

subject to the constraint $0 \le x \le \min\{X,\ W/\varepsilon\}$, and must satisfy the canonical equations (5) and

(12) $$\dot{\lambda}(t) = r\lambda(t) - \gamma x(t),$$

as well as the transversality condition:

(13) $$\lim_{t\to+\infty} e^{-rt} H(t) = 0.$$

Variable λ is the current-value shadow price of the groundwater stock.

Finally, a state-control pair $(s(t), x(t))$ satisfying the above necessary conditions is the solution to (P.1) if it also satisfies the Mangasarian sufficient condition (Caputo, 2005, theorem 14.4)—i.e., for any admissible pair $(\tilde{s}(t), \tilde{x}(t))$:

(14) $$\lim_{t\to+\infty} e^{-rt}\lambda(t)\left(s(t) - \tilde{s}(t)\right) \le 0.$$

A widely used strategy for solving the bang-bang type of problem is to identify the switch times when the control variable jumps from one state to another (e.g., Caputo, 2005, chap. 3). To begin, we solve the maximization of the linear Hamiltonian to obtain:

(15) $$x(t) = \begin{cases} \min\{X, W/\varepsilon\} & \text{if } \gamma(\bar{s} - s(t)) + \kappa\lambda(t) < b\varepsilon, \\ 0 & \text{if } \gamma(\bar{s} - s(t)) + \kappa\lambda(t) \ge b\varepsilon. \end{cases}$$

We define an auxiliary variable as:

$$c = \gamma(\bar{s} - s) + \kappa\lambda, \tag{16}$$

where c, $\gamma(\bar{s} - s)$, and $\kappa\lambda$ are the marginal (total) economic cost, marginal pumping cost, and marginal user cost of extracted groundwater, respectively. By control path (15), the optimal rate of extraction is the smaller of the agronomic and hydrological constraint if marginal revenue, $b\varepsilon$, exceeds marginal economic cost, c, and is zero otherwise.

Given control path (15), the state path, $s(t)$, is easily understood. Whenever irrigated farming is practiced, the water table declines at a constant rate of $\kappa \min\{X, W/\varepsilon\}$; otherwise, it remains constant. The key to solving (P.1) now becomes understanding the dynamic behavior of the costate variable $\lambda(t)$. As shown in appendix B, $\lambda(t)$ is nonnegative, i.e., $\lambda(t) \geq 0$, $\forall\, t \geq 0$. This makes economic sense in that the price of a freely disposable good should always be nonnegative.

Differentiating equation (16) with respect to time and substituting from equations (5) and (12) yields $\dot{c} = \gamma r \lambda$. The above lemma therefore implies that marginal economic cost $c(t)$ is nondecreasing in time. Intuitively, this is because the water table can never rise with a zero recharge rate, and therefore pumping costs will never decrease. Further, we can rule out the case in which $c(t) < b\varepsilon$ for all $t \geq 0$, because if that is the case, $x(t) = \min\{X, W/\varepsilon\} > 0$ for all $t \geq 0$, which implies the pumping cost will keep increasing as time goes on. Specifically,

$$\lim_{t \to +\infty} c(t) = +\infty,$$

a contradiction to $c(t) < b\varepsilon$ for all $t \geq 0$.

Now we are left with only two possibilities regarding the path of the marginal economic cost: either $c(t) \geq b\varepsilon$ for all $t \geq 0$ or $c(0) < b\varepsilon$ and $c(t) \geq b\varepsilon$ for some $t \in (0, +\infty)$. In the former case, the rate of extraction is zero throughout by control path (15). In the latter case, the farmer starts by pumping as much groundwater as the binding constraints allow until the water table declines in finite time to a threshold level at which $c(t) = b\varepsilon$. Because marginal economic cost $c(t)$ is nondecreasing in time, the farmer will never reverse back to irrigated farming. Hence, we can rewrite control path (15) as:

$$x(t) = \begin{cases} \min\{X, W/\varepsilon\} & \text{if } t < T. \\ 0 & \text{if } t \geq T, \end{cases} \tag{17}$$

where $T \in [0, +\infty)$. If $T = 0$, we have $c(t) \geq b\varepsilon$ for all $t \geq 0$; it $T > 0$, $c(t) \geq b\varepsilon$ for $t \in [T, +\infty)$ and $c(t) < b\varepsilon$ for all $t \in [0, T)$.

Next we show that $\lambda(t) = 0$ for $t \geq T$, which will give us the terminal condition for $\lambda(t)$: $\lambda(T) = 0$. Equations (5) and (17) imply that $s(t) = s(T) < +\infty$ for all $t \geq T$. That is, after irrigation is abandoned, the water table will remain unchanged. Thus, there exists at least one admissible pair $(\tilde{s}(t), \tilde{x}(t))$ such that

$$\lim_{t \to +\infty} s(t) - \tilde{s}(t) > 0.$$

For example, let $\tilde{s}(t)$ be a state path associated with a control path in the form of (17) with the switch time $T' = T + \delta$, where δ is an arbitrarily small positive number; evidently,

$$\lim_{t \to +\infty} s(t) - \tilde{s}(t) = s(T) - \tilde{s}(T) > 0.$$

Therefore, in order for the sufficient transversality condition (14) to hold, we must have

$$\lim_{t \to +\infty} e^{-rt}\lambda(t) = 0,$$

which, because $\dot{\lambda}(t) = r\lambda(t)$ for all $t \geq T$, implies $\lambda(t) = 0$ for $t \geq T$.

With the terminal condition for the costate variable determined and the initial condition for the state variable given, we can solve canonical equations (5) and (12) in terms of switch time, T:

(18)
$$s(t) = \begin{cases} s^0 - \kappa t \min\{X, W/\varepsilon\} & \text{for } t < T, \\ s^0 - \kappa T \min\{X, W/\varepsilon\} & \text{for } t \geq T; \end{cases}$$

(19)
$$\lambda(t) = \begin{cases} \dfrac{\gamma}{r}(1 - e^{-r(T-t)}) \min\{X, W/\varepsilon\} & \text{for } t < T, \\ 0 & \text{for } t \geq T. \end{cases}$$

Substituting $s(T)$ and $\lambda(T)$ from equations (18) and (19) into the cost equation (16) yields $c(T) = \gamma(\bar{s} - s^0 + \kappa T \min\{X, W/\varepsilon\})$. If $T = 0$, we have $c(T) = \gamma(\bar{s} - s^0)$, which, given equation (15) and $\lambda(T) = 0$, implies $\gamma(\bar{s} - s^0) \geq b\varepsilon$. This means irrigated farming is no more profitable than dryland farming at the initial period, and the latter will be practiced throughout. If $T > 0$, we have $c(T) = b\varepsilon$, i.e.,

$$T = \frac{s^0 - (\bar{s} - b\varepsilon/\gamma)}{\kappa \min\{X, W/\varepsilon\}}.$$

Thus, the switch time function can be written in the following compact form:

(20)
$$T = \max\left\{0, \frac{s^0 - (\bar{s} - b\varepsilon/\gamma)}{\kappa \min\{X, W/\varepsilon\}}\right\}.$$

Using Gisser and Sanchez's (1980) results, Koundouri (2004b) points out that when the slope of inverse water demand is close to infinity, myopic competitive extraction leads to an outcome which is close to the socially optimal. An inspection of equations (17), (18), and (20) against equations (6), (7), and (8) shows this result can be extended to the limiting case where the slope is infinity. Myopic competitive extraction leads exactly to the socially optimal outcome. This result can be better understood with a closer look at the path of the costate variable, $\lambda(t)$. The marginal user cost, $\kappa\lambda$, represents the externalities that individual resource users fail to account for and which constitute the driving force for the welfare loss arising from competitive extraction. In the case of perfectly inelastic demand, however, it does not matter whether or not this variable is taken into consideration. Specifically, equation (15) implies that so long as extraction is profitable, the rate of extraction is fixed at a level determined solely by the binding constraint and independent of the value of the marginal user cost. Although the equation suggests marginal user cost may affect the extraction path at the switch time, it becomes zero at that moment because the groundwater stock is no longer of value, i.e., $\lambda(T) = 0$ (see figure 1).

Since the model we present here is different from most existing groundwater models in the mathematical structure, it is instructive to conduct a comparative dynamics analysis of the solution path. Specifically, we examine how the extraction path and value of the groundwater resource will change in response to changes in a number of exogenous variables, including irrigation efficiency ε, marginal physical product b, unitary pumping cost γ,

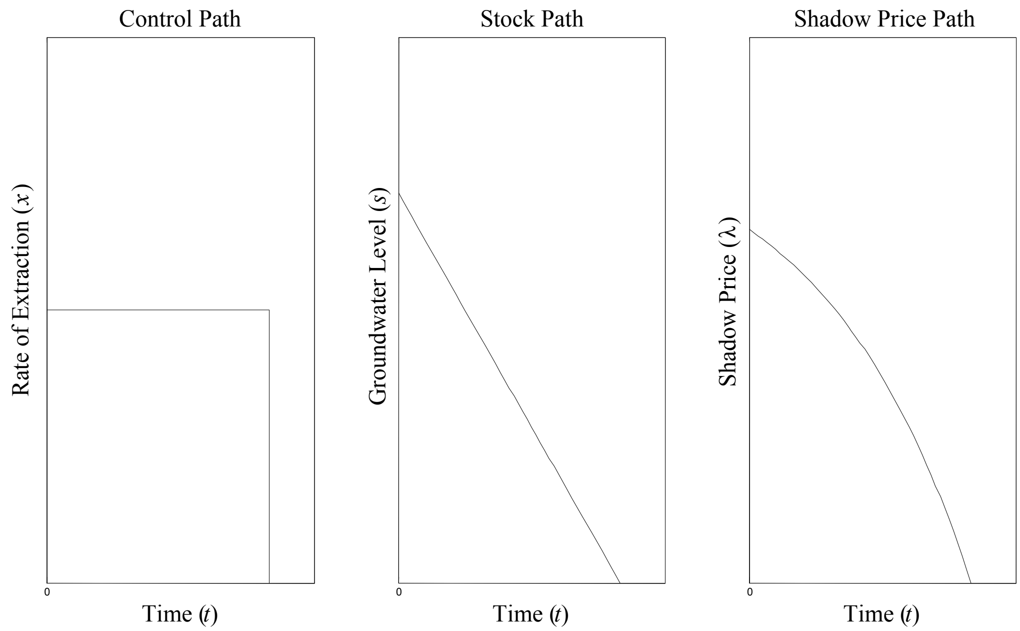

Figure 1. The control, stock, and shadow price paths in the representative-farm model

hydrologic parameter κ, agronomic constraint *W*, well yield capacity *X*, and farmland surface elevation $\bar{s}$.

Inspection of the switch time function (20) suggests irrigation efficiency improvements prolong the aquifer's usable life for irrigation (when $s^0 > \bar{s} - b\varepsilon / \gamma$), or may induce a switch from dryland to irrigated farming (when $s^0 = \bar{s} - b\varepsilon / \gamma$). The value of the groundwater resource therefore increases with irrigation efficiency if irrigated farming is ever an economical choice (see appendix B).

Similarly, any biotechnical progress that improves the marginal productivity of applied water, *b*, permits profitable irrigation at lower groundwater levels, prolonging the resource stock's usable life and boosting its total economic value. Additionally, the same can be said for an increase in crop price because it behaves in the model identically to *b*. Better farm management strategies or varietal improvements that relax biological constraint *W* can trigger more intensive irrigation and therefore shorten the usable life of the aquifer. This amounts to selling the groundwater at a faster rate, which, because of the positive discount rate, will increase the total value of the resource stock. Well yield capacity *X* behaves the same as does the agronomic constraint. If it is a binding constraint, then the higher it is, the more intensive will be the irrigation observed—and the more valuable the groundwater resource will be.

The longer the distance, $\bar{s} - s^0$, between the land surface elevation and the initial level of water table, the higher the pumping cost to begin with, the shorter the aquifer's usable life, and the lower its total economic value. An aquifer with higher storativity or larger covered area, which is negatively related to κ, is more valuable to the groundwater user because higher storativity allows the water table to decline and the pumping cost to increase more slowly. Finally, more efficient pumps or lower energy price (both lower the value of γ) permit profitable irrigation at a deeper water table, and thus increase the value of the groundwater stock.

The Two-Farm Model

The representative farm model is appropriate only when groundwater users are short-sighted, as if they are completely unaware of the commonality of the groundwater resource. Additionally, by way of aggregation, it precludes an understanding of productivity heterogeneity's effect on resource use efficiency. To overcome these limitations, we consider a model of two farms that may differ in water use productivity and that may take strategic actions in view of the dynamic interactions between them. We preserve the notation in the previous section and distinguish between the two farms by subscript $i = 1, 2$. The dynamics of the water table become:

$$\dot{s}(t) = \kappa \sum_{i=1}^{2} x_i(t), \tag{21}$$

where x_i is farm i's rate of extraction. The drawdown rate of the water table is linear in the sum of the two farms' extraction rates.

For generality, parameters a, b, and W in the production function, irrigation efficiency ε, elevation of the farmland surface $\overline{s}$, and well-yield capacity X are each allowed to vary between the two farms, but unit pumping cost γ is assumed to be the same between them. This is a fairly innocuous assumption, as pumping costs are determined by energy price and the engineering properties of the pump, which tend to be similar for farms within a groundwater basin.

Competitive Extraction

Consider first the two farms' Nash equilibrium extraction paths. Specifically, we assume each farm chooses the optimal path of extraction in order to maximize the discounted present value of profits with respect to the other's optimal extraction path. Formally, the two farms' extraction problem can be expressed as:

$$\text{(P.2.1)} \qquad \max_{x_i(\cdot)} \int_0^{\infty} e^{-rt} \Big[a_i + \Big[b_i \varepsilon_i - \gamma\big(\overline{s}_i - s(t)\big) \Big] x_i(t) \Big] dt$$

$$\begin{aligned} \text{s.t.:} \quad & \dot{s}(t) = -\kappa \sum_{i=1}^{2} x_i(t), \\ & s(0) = s^0, \\ & 0 \le x_i(t) \le Z_i = \min\{X_i, W_i / \varepsilon_i\}, \\ & i = 1, 2. \end{aligned}$$

The current-value Hamiltonian is written as:

$$H_i = a_i + \Big[b_i \varepsilon_i - \gamma(\overline{s}_i - s) - \kappa\lambda_i \Big] x_i - \kappa\lambda_i x_j, \quad i, j = 1, 2, \text{ and } i \ne j, \tag{22}$$

where the costate variable, λ_i, represents the shadow price of the groundwater resource to farm i.

Maximization of equation (22) with respect to the constraint set $0 \le x_i \le Z_i$ yields the necessary condition:

(23)
$$x_i(t) = \begin{cases} Z_i & \text{if } c_i < b_i \varepsilon_i, \\ 0 & \text{if } c_i \geq b_i \varepsilon_i, \end{cases} \quad i = 1, 2,$$

where $c_i = \gamma(\bar{s}_i - s) + \kappa\lambda_i$ is the marginal economic cost, which comprises the marginal pumping cost $\gamma(\bar{s}_i - s)$ and the marginal user cost $\kappa\lambda_i$. The canonical equations are (21) and

(24)
$$\dot{\lambda}_i(t) = r\lambda_i(t) - \gamma_i x_i(t).$$

The necessary and sufficient transversality conditions are omitted here. As in the representative-farm model, they guarantee that the maximum of the Hamiltonian (23) can be written as:

(25)
$$x_i(t) = \begin{cases} Z_i & \text{for } t < T_i, \\ 0 & \text{for } t \geq T_i, \end{cases}$$

and $\lambda_i(t) = 0, \forall t \geq T_i$ for $i = 1, 2$. Further, one can show that $T_i \in [0, +\infty)$ is determined by the following complementary slackness condition:

(26)
$$\begin{aligned} & T_i \geq 0, \\ & \gamma\left(\bar{s} - s(T_i)\right) - b_i \varepsilon_i \geq 0, \\ & T_i\left[\gamma\left(\bar{s}_i - s(T_i)\right) - b_i \varepsilon_i\right] = 0, \\ & i = 1, 2. \end{aligned}$$

When $T_i > 0$, we have $\gamma(\bar{s} - s(T_i)) - b_i\varepsilon_i = 0$. Since water level $s(t)$ is nonincreasing in time, this implies $\gamma(\bar{s} - s(t_i)) - b_i\varepsilon_i < 0$ for all $t < T_i$. That is, farm i practices irrigated farming until the water table declines to the threshold level where irrigated farming becomes equally as profitable as dryland farming. When the marginal pumping cost exceeds the marginal revenue of applied water at the initial state, i.e., $\gamma(\bar{s} - s^0) > b_i\varepsilon_i$, then dryland farming is always more profitable than irrigated farming, and thus is adopted throughout, i.e., $T_i = 0$.

For ease of notation, let $h_1 = \bar{s}_1 - b_1\varepsilon_1 / \gamma$ and $h_2 = \bar{s}_2 - b_2\varepsilon_2 / \gamma$. Assume, without loss of generality, $h_1 \geq h_2$; that is, farm 1 is less than or equally as productive as farm 2. As discussed earlier, in the representative-farm model, the productivity difference is due to differences in technological and hydrological conditions facing the two farms. It follows immediately from this assumption and (26) that $s(T_1) \geq s(T_2)$; that is, the less productive farm stops irrigation at a higher water level than does the more productive farm. Specifically, the former stops irrigation earlier than does the latter ($T_1 \leq T_2$). Integrating equation (21) and substituting from equation (25) yields the path of the water table:

(27)
$$s(t) = \begin{cases} s^0 - \kappa t(Z_1 + Z_2) & \text{for } t \in [0, T_1), \\ s^0 - \kappa T_1 Z_1 - \kappa t Z_2 & \text{for } t \in [T_1, T_2), \\ s^0 - \kappa T_1 Z_1 - \kappa T_2 Z_2 & \text{for } t \in [T_2, +\infty). \end{cases}$$

Given the terminal condition $\lambda_i(T_i) = 0$, the current-value shadow price can also be solved in terms of the switch times:

(28)
$$\lambda_i(t) = \begin{cases} \dfrac{\gamma}{r} Z_i(1 - e^{-r(T_i - t)}) & \text{for } t \in [0, T_i), \\ 0 & \text{for } t \in [T_i, +\infty), \end{cases} \quad i = 1, 2.$$

It remains to solve for the Nash equilibrium switch times for the two farms. Substituting for $s(T_i)$ from (27) into (26), we obtain:

$$T_i = \max\{0, \tau_i\}, \quad i = 1, 2, \tag{29}$$

where

$$\tau_1 = \frac{s^0 - h_1}{\kappa(Z_1 + Z_2)}, \tag{30}$$

$$\tau_2 = \frac{s^0 - \kappa\tau_1 Z_1 - h_2}{\kappa Z_2}. \tag{31}$$

Rearranging terms in (30) and (31) and subtracting (30) from (31) yields the difference between the two farms' switch times:

$$\tau_2 - \tau_1 = \frac{h_1 - h_2}{\kappa Z_2}. \tag{32}$$

Equations (27) and (30) indicate that farm 1 abandons irrigation when the water table declines to its threshold level, $s(\tau_1) = h_1$, and that the switch time is the amount of time needed for the water table to decline from the initial level, s^0, to the critical level, h_1, at a constant rate, $\kappa(Z_1 + Z_2)$. From equation (32), after farm 1 abandons irrigation, farm 2 will continue irrigation at the constant rate of Z_2 until the water table reaches its threshold level h_2.

The above description of the Nash equilibrium extraction paths sounds much like the behavior of two myopic farmers. To verify, all we need to do is exclude the marginal user cost from the total economic cost, precisely in light of the definition of myopic behavior. Specifically, we let $c_i = \gamma(\overline{s}_i - s)$ instead of $c_i = \gamma(\overline{s}_i - s) + \kappa\lambda_i$ in equation (23). The dropped term ($\kappa\lambda_i$) is the marginal user cost of extracted groundwater, representing the externalities that the myopic farmers fail to consider in their decision making. Equation (23) implies the two farms will continue to pump water from the aquifer so long as the marginal pumping cost c_i remains below the marginal benefits of extraction. If irrigated farming is more profitable than dryland farming in the initial state, both farms will maintain their respective fixed pumping rates given in equation (23). Therefore, the water table will decline at the same rate as in the Nash equilibrium. The switch time for farm 1 is when the groundwater level reaches the threshold h_1, and for farm 2 threshold h_2, exactly as described in equations (30) and (31).

The coincidence of the myopic and Nash equilibrium paths again can be best understood by an examination of the behavior of the marginal user cost, $\kappa\lambda_i$. In the Nash equilibrium, the marginal user cost, $\kappa\lambda_i$, is positive when farm i's extraction rate is anchored by the binding constraints and becomes zero at the very moment the farm abandons irrigation. Although in a Nash equilibrium the farmers do take into account the intertemporal stock externality, no change can be made to accommodate that externality due to the binding constraint on the rate of extraction. In other words, even if the farmers recognize the interdependence between their water use decisions and between today's action and tomorrow's choice, their choice is fixed by the binding constraints. Thus, the best they can do is to behave as if they are unaware of the externalities.

The same pumping behavior described above will be observed in a Markov-Nash equilibrium. Unlike the myopic equilibrium, the Markov-Nash strategy does take full account of the intertemporal and inter-agent externalities. Unlike the Nash equilibrium in which the farms determine their respective paths of extraction at the beginning of the game, it assumes their extraction decisions are made solely based on the current state of the resource. As a matter of convenience, the decision rule usually is assumed to be stationary or invariant over time (e.g., Provencher and Burt, 1993). The Markov-Nash strategy implies that the Hamiltonian (22) should be written slightly differently: x_j is replaced with $x_j(s)$. The change reflects the assumption that each farm believes the other's decision is made on the basis of the current state of the resource, s. This change, however, will not alter the necessary conditions (21), (23), and (24). Obviously, it will not change the state equation (21). Condition (23) will remain unaltered because the Hamiltonian is linear and its slope is independent of $x_j(s)$. The costate equation (24) is derived from the general formula: $\dot{\lambda}_i(t) = r\lambda_i(t) - \partial H_i / \partial s$. This necessary condition holds everywhere except for the points of discontinuities of $x_i(t)$ (see Caputo, 2005, theorem 14.3). An inspection of equation (23) shows that the optimal rate of extraction is independent of the state of the groundwater except for the switch point, indicating $\partial H_i / \partial s = 0$ everywhere except for the switch time. Thus, equation (23) is also the costate equation in the Markov-Nash equilibrium. To complete the proof, the Markov-Nash equilibrium is the closed-loop form of the open-loop solution (25):

$$x_i(s) = \begin{cases} Z_i & \text{if } s > h_i, \\ 0 & \text{if } s \le h_i, \end{cases} \quad i = 1, 2.$$

The coincidence of the Nash and Markov-Nash equilibriums is a direct result of demand perfect inelasticity. The Markov-Nash equilibrium concept emphasizes a type of "strategic interaction" between groundwater users (Provencher and Burt, 1993). Since they know their extraction decisions depend on the resource stock, which can be affected by their individual extraction decisions, one user may in principle change the other's action to his or her own advantage by altering the rate of extraction. In our model, however, a change in the groundwater stock does not affect the farms' extraction rates because of the binding constraints. In other words, the chain of strategic interactions is broken by water demand-prefect inelasticity.

As shown by the above analysis, when water demand is perfectly inelastic due to binding technical or hydrologic constraints, a simple strategy of "pumping as much as is possible without wasting" is optimal relative to myopic, Nash, and Markov-Nash equilibrium concepts. This certainly is a helpful property, since the behavioral assumptions associated with those equilibrium concepts are difficult, if not impossible, to test empirically.

The Optimal

Socially optimal paths of extraction can be derived from a straight optimal control problem in which two farms jointly determine rates of extraction to maximize present value of total profits over an infinite horizon:

$$\text{(P.2.2)} \qquad \max_{\{x_i(\cdot)\}_{i=1,2}} \int_0^\infty e^{-rt} \sum_{i=1}^{2} \left[a_i + \left[b_i \varepsilon_i - \gamma\left(\overline{s}_i - s(t)\right) \right] x_i(t) \right] dt$$

$$\text{s.t.:} \quad \dot{s}(t) = -\kappa \sum_{i=1}^{2} x_i(t),$$

$$s(0) = s^0,$$

$$0 \le x_i \le Z_i = \min\{X_i, W_i / \varepsilon_i\},$$

$$i = 1, 2.$$

The current-value Hamiltonian is:

$$H = \sum_{i=1}^{2} a_i + \left[b_i \varepsilon_i - \gamma(\bar{s} - s) - \kappa\lambda \right] x_i \,, \tag{33}$$

where λ is the costate variable representing the social shadow price of the groundwater stock.

Maximization of the Hamiltonian subject to the constraint sets $0 \le x_i \le Z_i$ yields:

$$x_i(t) = \begin{cases} Z_i & \text{if } c_i < b_i \varepsilon_i \,, \\ 0 & \text{if } c_i \ge b_i \varepsilon_i \,, \end{cases} \quad i = 1, 2, \tag{34}$$

where $c_i = \gamma(\bar{s}_i - s) + \kappa\lambda$ is the marginal economic cost, comprising marginal pumping cost, $\gamma(\bar{s} - s)$, and marginal user cost, $\kappa\lambda$. Note that equation (34) differs from equation (25) in that marginal user cost in cooperative model $\kappa\lambda$ is different from (and presumably larger than) those in non-cooperative model $\kappa\lambda_i$. The canonical equations are (21) and

$$\dot{\lambda}(t) = r\lambda(t) - \gamma \sum_{i=1}^{2} x_i(t). \tag{35}$$

We use a star notation to distinguish the solution paths of the state, costate, and control variables in the current model from those in the competitive extraction model. As before, the transversality conditions permit us to rewrite the maximum of the Hamiltonian as:

$$x_i^*(t) = \begin{cases} Z_i & \text{for } t < T_i^*, \\ 0 & \text{for } t \ge T_i^*, \end{cases} \quad i = 1, 2, \tag{36}$$

where T_i^* is determined by the following complementary slackness condition:

$$\begin{aligned} & T_i^* \ge 0, \\ & \gamma\left(\bar{s} - s(T_i^*)\right) + \kappa\lambda(T_i^*) - b_i \varepsilon_i \ge 0, \\ & T_i^* \left[\gamma\left(\bar{s}_i - s(T_i^*)\right) + \kappa\lambda(T_i^*) - b_i \varepsilon_i \right] = 0, \\ & i = 1, 2. \end{aligned} \tag{37}$$

Farm 1 is assumed to be less productive than farm 2, whereby the former's threshold water table is higher than the latter's, i.e., $h_1 \ge h_2$. As in competitive extraction, we can show that the farm with less favorable irrigation conditions will abandon irrigation earlier, i.e., $T_1^* \le T_2^*$. Similarly, one can arrive at $\lambda(t) = 0, \forall\, t \ge T_2^*$ through the sufficient transversality condition. With the terminal condition $\lambda(T_2) = 0$, costate equation (35) can be solved in terms of switch times. Integrating equation (35) and substituting for $x_i^*(t)$ from equation (36) into equation (35), we obtain the path of the social shadow price:

$$\lambda^*(t) = \begin{cases} \dfrac{\gamma}{r} \displaystyle\sum_{i=1}^{2} Z_i (1 - e^{-r(T_i^* - t)}) & \text{for } t \in [0, T_1^*), \\ \dfrac{\gamma}{r} Z_2 (1 - e^{-r(T_2^* - t)}) & \text{for } t \in [T_1^*, T_2^*), \\ 0 & \text{for } t \in [T_2^*, +\infty). \end{cases} \tag{38}$$

Canonical equation (21) can be solved with respect to switch times T_i^* as:

$$(39)\qquad s^*(t)=\begin{cases} s^0-\kappa t(Z_1+Z_2) & \text{for } t\in[0,T_1^*),\\ s^0-\kappa T_1^* Z_1-\kappa t Z_2 & \text{for } t\in[T_1^*,T_2^*),\\ s^0-\kappa T_1^* Z_1-\kappa T_2^* Z_2 & \text{for } t\in[T_2^*,+\infty). \end{cases}$$

Finally, substitute $\lambda^*(T_i^*)$ and $s^*(T_i^*)$, respectively, from equation (38) and equation (39) into equation (37) to obtain the optimal switch times:

$$(40)\qquad T_i^*=\max\{0,\tau_i^*\},\quad i=1,2,$$

where the following two equations are solved by τ_i^*:

$$(41)\qquad \tau_1^*=\frac{s_0-h_1-\dfrac{\kappa}{r}Z_2(1-e^{-r(\tau_2^*-\tau_1^*)})}{\kappa(Z_1+Z_2)},$$

$$(42)\qquad \tau_2^*=\frac{s_0-\kappa\tau_1^* Z_1-h_2}{\kappa Z_2}.$$

Rearranging terms in (41) and (42) and subtracting the former from the latter yields:

$$(43)\qquad \tau_2^*-\tau_1^*=\frac{h_1-h_2}{\kappa Z_2}+\frac{1-e^{-r(\tau_2^*-\tau_1^*)}}{r}.$$

One can verify in equation (43) that the socially optimal switch times of the two farms equalize if and only if they are equally productive; that is, $\tau_1^*=\tau_2^*$ if and only if $h_1=h_2$. Identical farms are sufficient but not necessary for this condition to hold.

The control and state paths (36) and (39) in the cooperative model look almost identical to their respective counterparts (25) and (27) in the competitive model. The only difference lies in the different switch times of the two farms, as can be seen by contrasting (30) with (41) and (31) with (42). Specifically, the difference for farm 1 is:

$$(44)\qquad \tau_1-\tau_1^*=\frac{\lambda(T_1^*)}{\kappa(Z_1+Z_2)}=\frac{Z_2(1-e^{-r(\tau_2^*-\tau_1^*)})}{r(Z_1+Z_2)}\geq 0,$$

where the equality holds when irrigation is simultaneously abandoned by the two farms ($\tau_1^*=\tau_2^*$) or, equivalently, when the two farms are equally productive ($h_1=h_2$). The difference in switch time for farm 2 is easily obtained by subtracting equation (42) from equation (31):

$$(45)\qquad \tau_2-\tau_2^*=-\frac{Z_1(1-e^{-r(\tau_2^*-\tau_1^*)})}{r(Z_1+Z_2)}\leq 0,$$

where the equality holds when $\tau_1^*=\tau_2^*$ or $h_1=h_2$.

Unless the two farms face equally favorable irrigation conditions ($h_1=h_2$), competitive extraction leads the less productive farm to extract groundwater for a longer period than is socially optimal. Consequently, the more productive farm is left with an amount that is less than optimal. When the two farms are equally productive ($h_1=h_2$), $\tau_1^*=\tau_2^*=\tau_1=\tau_2$; that is,

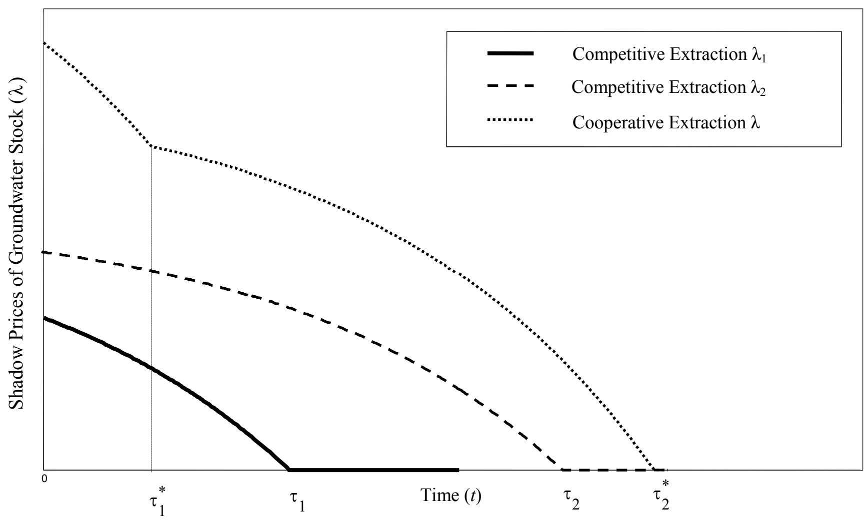

Figure 2. Shadow price paths and switch times in the two-farm model

competitive farms pump as much as their respective hydrological and technological constraints allow, then switch to dryland farming simultaneously. However, this is exactly the best they could do if they had cooperated—the CPR dilemma is avoided when water demand is perfectly inelastic and when groundwater users are equally productive.

Inspecting the path of the costate variables can help us better understand these results. Figure 2 illustrates the competitive and social shadow price paths and switch times in the presence of heterogeneity ($h_1 > h_2$). It shows that the social and private difference in the switch time of each farm emanates from the discrepancy between the social and private shadow values of the groundwater resource. Under competitive extraction, farm 1 stops irrigation only when the groundwater resource is of no value to itself ($\lambda_1(T_1) = 0$); therefore, the marginal economic cost consists only of the marginal pumping cost at that switch time. This finding implies farm 1 abandons irrigation when the marginal pumping cost equals the marginal product value ($\gamma(\bar{s}_1 - s(T_1)) - b_1\varepsilon_1 = 0$). In the case of cooperative extraction, the social shadow value of the resource is zero only if farm 2 quits irrigation ($\lambda(T_2^*) = 0$), and is positive when farm 1 quits irrigation ($\lambda(T_1^*) > 0$). A social planner would force farm 1 to quit irrigation whenever the marginal economic cost (composed of the pumping and user costs) equals the marginal product value ($\gamma(\bar{s}_1 - s(T_1^*)) + \kappa\lambda(T_1^*) - b_1\varepsilon_1 = 0$). Because of the positive marginal user cost, $\lambda(T_1^*)$, we have $T_1^* < T_1$.

More importantly, $\lambda(T_1^*)$ measures the extra benefit that slightly more extraction by farm 2 can generate at time T_1^* when farm 1 stops irrigation, which is the opportunity cost of slightly more extraction by farm 1 at time T_1^*. We can rewrite the equation for farm 1's switch time as:

$$\kappa\lambda(T_1^*) = b_1\varepsilon_1 - \gamma\left(\bar{s}_1 - s(T_1^*)\right). \tag{46}$$

The right-hand side of the equation is the extra benefit of a slightly further extraction by farm 1, and the left-hand side is the opportunity cost of this extra unit of extraction. Thus, the equation provides a familiar arbitrage interpretation. The optimal switch time for the less productive farm is when a slightly further extraction by either farm generates the same amount

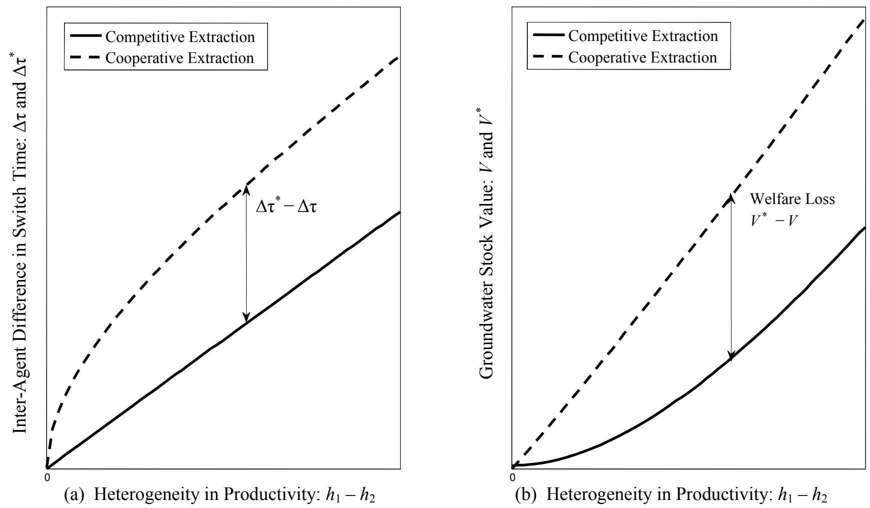

Figure 3. Differences in switch time and welfare loss each increase with the degree of heterogeneity in the two-farm model

of extra benefits and no reallocation of groundwater between the two can further improve their total welfare. The inefficiency caused by heterogeneity falls into the category of assignment problems in Ostrom, Gardner, and Walker's (1994) taxonomy of CPR problems. In the absence of well-defined, tradable property rights to groundwater units, assignment problems arise because a coordinating mechanism is lacking to allocate the resource in accordance with productivity. Simulation analyses have found that heterogeneity tends to exacerbate the CPR problem (Laukkanen and Koundouri, 2006; Worthington, Burt, and Brustkern, 1985). The mechanisms of such an effect remain unclear, however, because of the difficulty in separating it from the stock externalities in those simulation models. Here, the assumption of water demand perfect inelasticity serves as a filter to purge out stock externalities whereby a clear understanding of the problem becomes possible.

Next, we examine the relationship between the degree of heterogeneity, $h_1 - h_2$, and welfare loss, $\Delta V = V^* - V$, where V^* is the value function in the cooperative model (P.2.2), and V is the summation of the two farms' value functions in the competitive model (P.2.1). In appendix D, we show that:

$$\text{(47)} \qquad \frac{\partial \Delta V}{\partial h_1} = \frac{e^{-r\tau_1^*} - e^{-r\tau}}{r}\gamma Z_1 > 0,$$

$$\text{(48)} \qquad \frac{\partial \Delta V}{\partial h_2} = \frac{e^{-r\tau_2^*} - e^{-r\tau_2^*}}{r}\gamma Z_2 < 0.$$

That is, efficiency loss, ΔV, increases when the productivity of less productive farm 1 decreases or when the productivity of more productive farm 2 increases. Thus, efficiency loss increases in the degree of heterogeneity, $h_1 - h_2$. Figures 3a and 3b illustrate this property. As shown in figure 3a, the larger the difference in productivity, the larger the gap between the optimal and competitive inter-agent difference in switch time. This implies the presence of a more serious assignment problem as evidenced by greater efficiency loss shown in figure 3b.

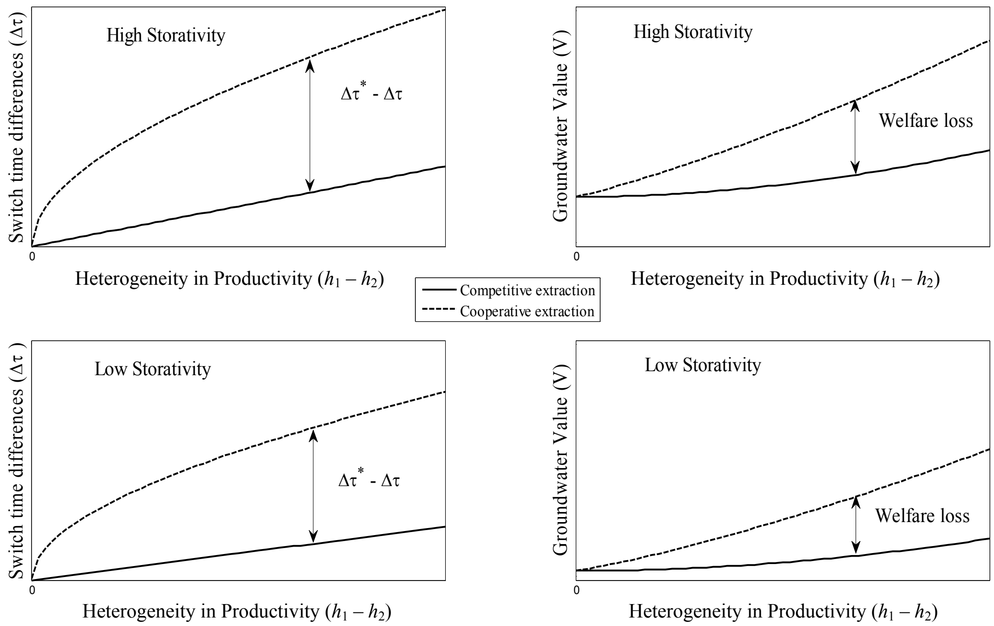

Figure 4. Increased storativity augments the magnitude of heterogeneity-induced welfare loss in the two-farm model

In particular, in the absence of productivity differences, the two farms' competitive switch times equalize, which is socially optimal.

We now analyze the influences of some exogenous parameters on the magnitude of the heterogeneity-induced efficiency loss. Gisser and Sanchez (1980) have found that efficiency loss under myopic, competitive extraction tends to be small for aquifers with large storativity and covered area. In our model, aquifer storativity (S) and area (A) are negatively associated with parameter κ. An inspection of equations (32), (44), and (47) reveals that a reduction in κ, namely an increase in storage capacity, SA, will increase $\tau_1 - \tau_1^*$ and $\tau_2^* - \tau_2$. An increase in $\tau_1 - \tau_1^*$, in turn, will increase $d\Delta V/dh_1$, and an increase in $\tau_2^* - \tau_2$ will decrease $d\Delta V/dh_2$ (since the exponential function e^{-x} is monotonically decreasing). Overall, an increase in aquifer storage capacity augments the efficiency loss induced by heterogeneity. Figure 4 provides an illustration of this effect. In an aquifer with larger storativity, a given degree of heterogeneity leads to a larger gap between the optimal and competitive inter-agent difference in switch time, and thus greater efficiency loss, than in an aquifer with smaller storativity.

The efficiency effect of storage capacity in our model contrasts with that predicted by Gisser and Sanchez (1980). Their model assumes that symmetric groundwater users with myopic behavior maximize own current benefits, taking no account of the present extraction's influences on future pumping costs. For illustrative purposes, we break down the stock externality into two stages. The first stage is when a change in the current extraction rate causes a change in the pumping lift. The second stage is when a change in the pumping lift results in a change in the future extraction rate because of higher pumping costs. When aquifer storage capacity (SA) is large relative to extraction rate, the resulting decline of water table and increase in pumping lift tend to be insignificant. This means the first stage of the stock externality is weak. When the demand is inelastic, the future extraction rate is irresponsive to changes in pumping lift, indicating the second stage of the stock externality is weak. Since the difference between the socially optimal and competitive extraction paths is determined by whether the stock externality is taken into consideration, that difference tends to be small when the stock externality is weak, either because of high storage capacity or because of water demand inelasticity. In particular, when demand is perfectly inelastic, the second stage of the stock externality is completely broken and there is no stock externality at all. This explains why the competitive extraction is optimal in the case of demand perfect inelasticity and homogeneous users.

As described earlier, the CPR problem in our model is an assignment problem caused by heterogeneity in productivity, which is measured by the difference in the threshold water level, $h_1 - h_2$. This is because of the lack of a water-rationing scheme that assigns groundwater on the basis of user productivity. At a given level of productivity difference, $h_1 - h_2$, higher storage capacity means that more economic value is at stake when groundwater is not allocated in accordance with productivity. It might be intuitively useful to consider storage capacity as a measure of the "quality" of the groundwater resource, while the "quantity" is measured by the thickness of the aquifer. For a given quantity of a resource, misallocation will incur greater loss if the quality of that resource is higher.[4]

Conclusions and Policy Implications

In this study, we have examined the common-pool resource problems in using a nonrenewable groundwater resource for agricultural irrigation when water demand is perfectly inelastic. Our analysis complements the existing groundwater use theory in several ways. Water demand perfect inelasticity has traditionally been treated as a special case with little empirical relevance. The accumulated evidence that water demand is generally inelastic, and sometimes perfectly inelastic, calls for a careful treatment of this extreme case. Many real-world situations likely exhibit water demand inelasticity. Examples include situations where groundwater supplies are in deficit because of limited well yield, where groundwater is abundant but yield has tapped into a plateau in the presence of some other limiting factor, where irrigation efficiency improvements are difficult either because the existing system is already close to perfectly efficient or because farmers cannot afford more efficient and expensive systems, and when crop choices are limited by climatic and soil factors. Our analysis provides a comprehensive treatment of the CPR dilemma in these situations and sheds light on some predictions of previous models. By examining the limiting case of water demand perfect inelasticity, our analysis reveals why welfare losses caused by stock externalities tend to be small when water demand is inelastic. This limiting assumption also serves as a filter to remove the stock externality so that the heterogeneity effect can be analyzed.

Our analysis leads us to predict that the optimal competitive extraction strategy is one of pumping as much as possible without wasting, regardless of whether groundwater users behave myopically or strategically in the Nash and Markov-Nash senses. We also predict that competitive extraction creates no welfare loss at all when groundwater users are symmetric,

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[4] Most of the results in our nonrenewable groundwater extraction model can be extended to the case in which the aquifer receives positive recharge but the recharge rate is slower than the extraction rate. In that case, the optimal private extraction path is to extract as much as possible, and then switch to deficit irrigation where the rate of extraction equals the rate of recharge. This slight change in the steady state of the extraction path does not affect our results on the private and social efficiency comparisons. There is no reason to study the economics of groundwater extraction when the recharge rate is above the extraction rate, because no scarcity problem arises in such a case.

confirming that predictions by Gisser and Sanchez (1980) can be extended to the limiting case of demand perfect inelasticity. Further, symmetry is a sufficient but unnecessary condition for the above statement to hold—welfare loss will not occur as long as resource users are equally productive in using the groundwater resource to generate profits. We further predict that welfare loss arises in the presence of productivity heterogeneity and increases with the degree of heterogeneity. Cooperative extraction can eliminate that loss by creating incentives for the less productive user to stop extraction earlier than he or she would under competitive extraction; the problem is caused by the lack of a rationing scheme for allocating the resource on the basis of productivity. Finally, we predict that the magnitude of the heterogeneity-induced welfare loss is greater for aquifers with larger storage capacity, in stark contrast to the welfare effect of storage capacity driven by the stock externality (Gisser and Sanchez).

These results suggest more attention should be paid to water demand elasticity and user heterogeneity in groundwater management. Given the abundant evidence on water demand inelasticity and given the different types of CPR problems under elastic and perfectly inelastic water demand, an important point of departure for policy analysis should be an unambiguous test for the hypothesis of demand perfect inelasticity. When water demand is found to be inelastic, such that the stock externality is negligible, the focus of the research should be shifted to estimation of the degree of user heterogeneity in resource use efficiency. In regions where productivity levels are largely homogeneous, it is doubtful that a centralized, optimal control approach can improve producer welfare, and conservation measures proposed on the basis of the CRP dilemma are not justified. In regions where productivity varies among irrigators, welfare loss may arise, but the cause of that loss is not stock externality—rather, it is a misallocation of the resource among users with varying degrees of productivity.

A useful conceptual framework for solving such misallocation problems is the renowned Coase theorem. In particular, when water rights are well established and tradable, and irrigators are well informed about their productivity differences and the economic implications, bargaining and trading water rights could lead to an efficient outcome. For instance, more efficient users could pay less efficient users to abandon irrigation earlier than they would otherwise, so that purchased water could be used to generate more benefits than the payment. In that case, policy efforts should be aimed at establishing a functioning market for water rights and at supporting research which will improve public understanding of aquifer hydrology and economic benefits of trading water rights.

The model analyzed here has its own limitations, which require mention for future research in this area. Many empirical studies have found that producers may change cropping patterns and irrigation technologies in response to changes in water price. Like the majority of existing groundwater extraction models, ours assumes away crop choices for simplicity. Although our model considers the transition from irrigated to dryland farming, irrigation efficiency is taken as exogenously given. Also, for the purpose of comparison, we follow the "bathtub aquifer" assumption in the majority of groundwater use models, i.e., the aquifer's transmissivity is infinite. Recent work has begun to relax this unrealistic assumption to examine the CPR problem under finite transmissivity (e.g., Brozovic, Sunding, and Zilberman, 2003; Saak and Peterson, 2007). Future research will extend the current model to accommodate these choices and events.

[*Received March 2010; final revision received December 2010.*]

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Appendix A: The LRP Technology

For most crops, a linear relationship exists between relative yield loss and relative water deficit (Doorenbos and Kassam, 1979), such that:

(A1)
$$\left(1-\frac{Y_a}{Y_m}\right)=k_y\left(1-\frac{ET_a}{ET_m}\right),$$

where

Y_a = actual harvested yield,
Y_m = maximal harvested yield,
k_y = yield response factor,
ET_a = actual evapotranspiration, and
ET_m = maximal evapotranspiration.

By rearranging terms, actual harvested yield, Y_a, can be expressed as a linear function of actual evapotranspiration, ET_a:

(A2)
$$Y_a=(1-k_y)Y_m+k_y\frac{Y_m}{ET_m}ET_a.$$

Maximal harvested yield Y_m, yield response factor k_y, and maximal evapotranspiration ET_a are determined by soil quality, climate conditions, and genetic factors. Once the seed has been planted, the producer has no control over these factors—i.e., they are the parameters of the crop production function. Letting

$$a=(1-k_y)Y_m \text{ and } b=k_y\frac{Y_m}{ET_m},$$

equation (A2) can be rewritten as:

(A3)
$$Y_a=a+bET_a.$$

Since maximum yield is determined by the plant's varietal potential, the actual yield cannot exceed that maximum, i.e., $Y_a \le Y_m$. Adding this constraint to equation (A3) gives the linear response and plateau production function (1) in the text. □

Appendix B: Nonnegative Shadow Price

We assume that $\lambda(t)<0$ for some $t=\tau\ge 0$ and then derive a contradiction from the assumption. Because the rate of extraction, x, from equation (15) is nonnegative, an inspection of equation (12) suggests that $\lambda(\tau)<0$ implies $\lambda(t)<0$ for all $t\ge\tau$. Differentiating equation (16) with respect to time yields $\dot{c}=\kappa r\lambda$ and $\ddot{c}=\kappa r\dot{\lambda}=\kappa r(r\lambda-\gamma x)$. It follows that $\dot{c}(t)<0$ and $\ddot{c}(t)<0$ for $t\ge\tau$. This implies that

$$\lim_{t\to+\infty}c(t)<b\varepsilon,$$

which, because of equation (15), leads to:

$$\lim_{t\to+\infty} x(t) = \min\{X, W/\varepsilon\} > 0.$$

Because the water table is bounded above,

$$\lim_{t\to+\infty} e^{-rt}s(t) = 0.$$

Hence, the necessary transversality condition (13) reduces to:

$$\lim_{t\to+\infty} e^{-rt}\lambda(t) = 0.$$

Since equation (12) can be written as

$$\frac{d[e^{-rt}\lambda(t)]}{dt} = -\gamma x(t)e^{-rt},$$

$e^{-rt}\lambda(t)$ is nonincreasing in time. Since $\lambda(t) < 0$ for all $t \geq \tau$, we have

$$\lim_{t\to+\infty} e^{-rt}\lambda(t) < 0,$$

a contradiction. □

Appendix C: Comparative Dynamics in the Representative-Farm Model

This appendix provides the partial derivatives of the value function of problem (P.1) to support the discussion in the text on how the value of the groundwater resource changes in response to a change in exogenous variables. First, write the Lagrangean corresponding to the Hamiltonian:

$$\text{(A4)} \qquad L = a + \left[b\varepsilon - \gamma(\bar{s} - s) - \kappa\lambda\right]x + \mu_1(X - x) + \mu_2(W/\varepsilon - x) + \mu_3 x,$$

where μ_1, μ_2, $\mu_3 \geq 0$ are the Lagrangean multipliers associated with the constraints $x \leq X$, $x \leq W/\varepsilon$, and $x \geq 0$, respectively.

Integrating the maximized Lagrangean, the value function of problem (P.1) is:

$$\text{(A5)} \qquad V(\cdot) = \int_0^T \left[b\varepsilon - \gamma(\bar{s} - s) - \kappa\lambda\right]x + \mu_1(X - x) + \mu_2(W/\varepsilon - x) + \mu_3 x e^{-rt}\,dt + \int_0^\infty a e^{-rt}\,dt.$$

We first examine the response of the value function to irrigation efficiency improvements. By the envelope principle (Caputo, 2005, theorem 9.1):

$$\text{(A6)} \qquad \frac{dV}{d\varepsilon}(\cdot) = \frac{\partial V}{\partial \varepsilon}(\cdot) = \int_0^T \left(b\min\{X, W/\varepsilon\} - \mu_2 W/\varepsilon^2\right)e^{-rt}\,dt.$$

If $X < W/\varepsilon$, $\mu_2 = 0$, and therefore

$$\frac{dV}{d\varepsilon}(\cdot) \geq 0,$$

where the equality holds when $T = 0$. If $X > W/\varepsilon$, we need to solve for $\mu_2 > 0$ to determine the sign of equation (A6). Differentiating the Lagrangean (A4) with respect to x to obtain the optimizing condition for μ_2:

$$\text{(A7)} \qquad \mu_2 = b\varepsilon - \gamma(\bar{s} - s) - \kappa\lambda.$$

Substituting from (A7) into (A6) yields:

$$\text{(A8)} \qquad \begin{aligned} \frac{dV}{d\varepsilon}(\cdot) &= \int_0^T \left[bW/\varepsilon - \left(b\varepsilon - \gamma(\bar{s} - s) - \kappa\lambda\right)W/\varepsilon^2\right]e^{-rt}\,dt \\ &= \int_0^T \left[\left(\gamma(\bar{s} - s) + \kappa\lambda\right)W/\varepsilon^2\right]e^{-rt}\,dt \geq 0, \end{aligned}$$

where the equality holds when $T = 0$.

If $X = W/\varepsilon$, any increase in ε will lead to $\mu_2 = 0$, and then to

$$\frac{dV}{d\varepsilon}(\cdot) \geq 0.$$

Denoting by

$$\frac{dV}{d\varepsilon^+}(\cdot)$$

the directional derivative of V with respect to an increase in ε, we have:

$$\text{(A9)} \qquad \frac{dV}{d\varepsilon^+}(\cdot) \geq 0,$$

where the equality holds when $T = 0$. That is, the value of the groundwater resource appreciates with irrigation efficiency, if irrigation is ever economically feasible.

Similarly, one can show:

$$\text{(A10)} \qquad \frac{dV}{dW^+}(\cdot) = \int_0^T \frac{\mu_2}{\varepsilon} e^{-rt} dt = \begin{cases} = 0 & \text{if } X \leq W/\varepsilon, \\ \geq 0 & \text{if } X > W/\varepsilon; \end{cases}$$

$$\text{(A11)} \qquad \frac{dV}{dX^+}(\cdot) = \int_0^T \mu_1 e^{-rt} dt = \begin{cases} \geq 0 & \text{if } X < W/\varepsilon, \\ = 0 & \text{if } X \leq W/\varepsilon; \end{cases}$$

$$\text{(A12)} \qquad \frac{dV}{db}(\cdot) = \int_0^T \varepsilon \min\{X, W/\varepsilon\} e^{-rt} dt \geq 0;$$

$$\text{(A13)} \qquad \frac{dV}{d(\bar{s} - s_0)}(\cdot) = \int_0^T -\gamma \min\{X, W/\varepsilon\} e^{-rt} dt \leq 0;$$

$$\text{(A14)} \qquad \frac{dV}{d\gamma}(\cdot) = \int_0^T -\left(\bar{s} - s(t)\right) \min\{X, W/\varepsilon\} e^{-rt} dt \leq 0;$$

$$\text{(A15)} \qquad \frac{dV}{d\kappa}(\cdot) = \int_0^T -\lambda \min\{X, W/\varepsilon\} e^{-rt} dt \leq 0.$$

These results are interpreted in the text. □

Appendix D:
Comparative Dynamics in the Two-Farm Model

This appendix provides the formal derivation of the effects of the degree of productivity heterogeneity on the magnitude of the welfare loss caused by heterogeneity in the two-farm model. Specifically, we derive the marginal values of the productivity index,

$$\frac{dV^*}{dh_1} \text{ and } \frac{dV}{dh_1},$$

under cooperative and competitive extraction. The sign of the difference between these two derivatives, i.e.,

$$\frac{d\Delta V}{dh_1} = \frac{dV^*}{dh_1} - \frac{dV}{dh_1},$$

tells whether efficiency loss arising in competitive extraction expands or shrinks with an increase in h_1. Similarly, one can evaluate the efficiency effects of a reduction in h_2.

The value function $V^*(\cdot)$ for problem (P.2.2) represents the maximal benefits the two farms can jointly obtain from groundwater use. In order to take derivatives with respect to h_1 and h_2, we need to first rewrite the value function in terms of these indices as:

$$V^*(\cdot) = \frac{a_1 + a_2}{r} + \int_0^{\tau_1^*} e^{-rt} \sum_{i=1}^{2} \gamma\left(s(t) - h_i\right) Z_i \, dt + \int_{\tau_1^*}^{\tau_2^*} e^{-rt} \gamma\left(s(t) - h_2\right) Z_2 \, dt.$$

Invoking the envelope theorem, we have:

(A16) $$\frac{\partial V^*}{\partial h_1}(\cdot) = -\frac{1-e^{-r\tau_1^*}}{r}\gamma Z_1,$$

(A17) $$\frac{\partial V^*}{\partial h_2}(\cdot) = -\frac{1-e^{-r\tau_2^*}}{r}\gamma Z_2.$$

Similarly, the value function for problem (P.2.1) can be written as the sum of the two farms' respective value functions:

$$V^*(\cdot) = \frac{a_1+a_2}{r} + \int_0^{\tau_1} e^{-rt}\sum_{i=1}^{2}\gamma\left(s(t)-h_1\right)Z_i\,dt + \int_{\tau_1}^{\tau_2} e^{-rt}\gamma\left(s(t)-h_2\right)Z_2\,dt.$$

Applying the envelope theorem, its derivatives with respect to h_1 and h_2 are:

(A18) $$\frac{\partial V}{\partial h_1}(\cdot) = -\frac{1-e^{-r\tau_1}}{r}\gamma Z_1,$$

(A19) $$\frac{\partial V}{\partial h_2}(\cdot) = -\frac{1-e^{-r\tau_2}}{r}\gamma Z_2.$$

Subtracting (A18) from (A16) yields:

$$\frac{\partial \Delta V}{\partial h_1} = \frac{\partial V^*}{\partial h_1} - \frac{\partial V}{\partial h_1} = \frac{e^{-r\tau_1^*}-e^{-r\tau_1}}{r}\gamma Z_1 > 0 \quad (\text{since } \tau_1^* < \tau_1).$$

In other words, when h_2 is held constant, the welfare loss increases with h_1. Likewise, we can subtract (A19) from (A17) to obtain:

$$\frac{\partial \Delta V}{\partial h_2} = \frac{\partial V^*}{\partial h_2} - \frac{\partial V}{\partial h_2} = \frac{e^{-r\tau_2^*}-e^{-r\tau_2}}{r}\gamma Z_2 < 0 \quad (\text{since } \tau_2^* > \tau_2).$$

When h_1 is held unchanged, the welfare loss increases when h_2 decreases. Overall, welfare loss, ΔV, increases with the degree of heterogeneity, $h_1 - h_2$. □